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CHAOS IN A TWO-DEGREE OF FREEDOM DUFFING OSCILLATOR

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Abstract. *High dimensional dynamical systems have intricate behavior either on temporal or on spatial evolution properties. Nevertheless, most of the work on chaotic dynamics has been concentrated on temporal behavior of low-dimensional systems. This contribution reports on the chaotic response of a two-degree of freedom Duffing oscillator. Since the equations of motion are associated with a four-dimensional system, the analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, connected by a spring-dashpot system. With this configuration, it is possible to analyze the transmissibility of motion between the two oscillators.*

Keywords: *Chaos, Spatiotemporal chaos, Non-linear dynamics, Duffing oscillator.*

1. INTRODUCTION

Non-linear dynamics of mechanical systems presents some characteristics not observed in linear systems. As an example, one could mention chaotic motion where unpredictability and sensitivity on initial conditions are some important characteristics. The study of chaos considers proper mathematical and geometrical aspects. Therefore, new analytical, computational and experimental methods are developed to analyze the non-linear response of dynamical systems. Since these aspects usually consider geometrical approach, they introduce difficulties to describe systems with many degrees of freedom (Alligood *et al.*, 1997; Savi, 1997; Moon, 1992; Hilborn, 1994; Mullin, 1993; Ott, 1993; Kapitaniak, 1991; Wiggins, 1990; Schuster, 1989; Thompson & Stewart, 1986; Guckenheimer & Holmes, 1983). High dimensional dynamical systems have intricate behavior either on temporal or on spatial evolution properties. Nevertheless, most of the work on chaotic dynamics has been concentrated on temporal behavior of low-dimensional systems (Umberger *et al.*, 1989).

Many researches have been developed to study dynamical systems described by simple mathematical models. Despite the deceiving simplicity of these models, their nonlinear dynamic response may exhibit a number of interesting, complex behaviors. Mathematically, there are two kinds of dynamical models: *differential equations model*, which is continuous in time, and *map*, which describes the time evolution of a system by expressing its state as a function of its previous time. Hence, map is a dynamical system moving through time in discrete updates. One of the most important uses of maps is to assist in the study of a differential equation model (Alligood *et al.*, 1997). Duffing and van der Pol oscillators, non-linear pendulum and Lorenz system are some examples of classical dynamical systems described by differential equations model (Guckenheimer & Holmes, 1983). On the other hand, logistic and tent map are some of the one-dimensional maps while Henon and Ikeda map are some of the classical two-dimensional maps (Alligood *et al.*, 1997; Ott, 1993).

The Duffing oscillator has been used to model the non-linear dynamics of special types of mechanical and electrical systems. The differential equation that describes this oscillator has a cubic non-linearity, and it has been named after the studies of G. Duffing in the 1930s. A number of physical systems can be described using Duffing equation. As some examples, one could mention an electrical circuit with a non-linear inductor and the postbuckling vibrations of an elastic beam column under compressive loads.

The present contribution discusses the non-linear dynamics of a Duffing oscillator with two-degree of freedom. Special attention is given to chaotic motion and since the equations of motion are associated with a four-dimensional system, the analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, connected by a spring-dashpot system. With this assumption, it is possible to analyze the transmissibility of motion between the two oscillators. Some conclusions are made and may be used to understand the behavior of other dynamical system either with multiple degrees of freedom or continuous.

2. EQUATIONS OF MOTION

Consider a two-degree of freedom oscillator depicted in Fig.1. It consists of two masses, m_i (i = 1,2), supported by non-linear springs with stiffness K_i (i = 1,2,3) and linear dampers with coefficient c_i (i = 1,2,3). The system is harmonically excited by two forces $F_i = \delta_i \sin(\Omega_i t)$ (i = 1,2).

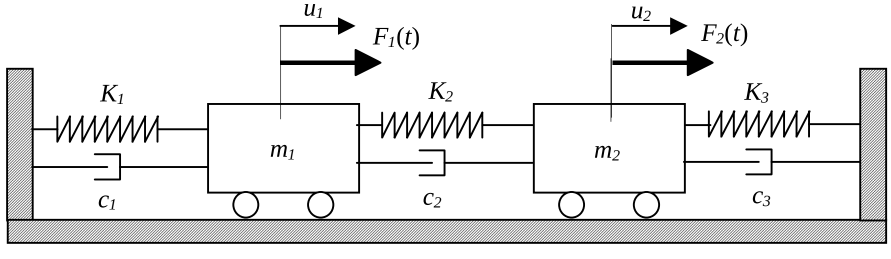

Figure 1 - Two-degree of freedom Duffing oscillator.

The non-linear spring behavior is described by considering that the restoring force is not linearly proportional to the displacement. The behavior of each spring is defined by the following function, where a cubic non-linearity is conceived,

$$K_i = K_i(u) = k_i u + a_i u^3 \tag{1}$$

The variable u represents the displacement associated with the spring; k_i and a_i are constants. By establishing the equilibrium of the system and assuming $y_0 = u_1$, $y_1 = \dot{u}_1$, $y_2 = u_2$ and $y_3 = \dot{u}_2$, the following dynamical system is written

$$\begin{aligned}
\dot{y}_0 &= y_1 \\
\dot{y}_1 &= (1/m_1)[F_1(t) - (c_1 + c_2)y_1 + c_2 y_3 - (k_1 + k_2)y_0 + k_2 y_2 - a_1 y_0^3 + a_2(y_2 - y_0)^3] \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= (1/m_2)[F_2(t) + c_2 y_1 - (c_2 + c_3)y_3 + k_2 y_0 - (k_2 + k_3)y_2 - a_2(y_2 - y_0)^3 - a_3 y_2^3]
\end{aligned} \tag{2}$$

The characterization of chaotic motion must be considered by some criterion, which establishes a quantitative definition of chaos. Lyapunov exponents are an acceptable criterion, which is used in this article by considering the algorithm proposed by Wolf *et al.* (1985).

In the following sections, numerical simulations of the forced response of the Duffing oscillator are discussed. In all simulations, one has taken $m_1 = m_2 = 1$, $k_1 = k_3 = -0.02$, $a_1 = a_3 = 1$, $c_1 = c_3 = 0.05$. It is also considered a harmonic excitation $F_i(t) = \delta_i \sin(\Omega_i t)$ ($i = 1,2$). Fourth order Runge-Kutta method is used to numerical integration and time steps less than $\Delta t = 2\pi / 600\Omega$ present good results.

3. LINEAR CONNECTION

In this Section, two Duffing oscillators, both with one-degree of freedom, connected with a linear spring-viscous damper system are discussed. Therefore, one considers $a_2 = 0$, and the parameters c_2 and k_2 are used to analyze the transmissibility of motion between the two oscillators. By considering $c_2 = k_2 = 0$, it is clear that there are two independent oscillators. The same frequency parameter is taken for both oscillators, $\Omega_1 = \Omega_2 = 1$, while two different forcing amplitudes are assumed: $\delta_1 = 7.5$ and $\delta_2 = 4$. The parameter $\delta_1 = 7.5$ causes chaotic motion on the first oscillator while $\delta_2 = 4$ results in a periodic motion (Fig.2).

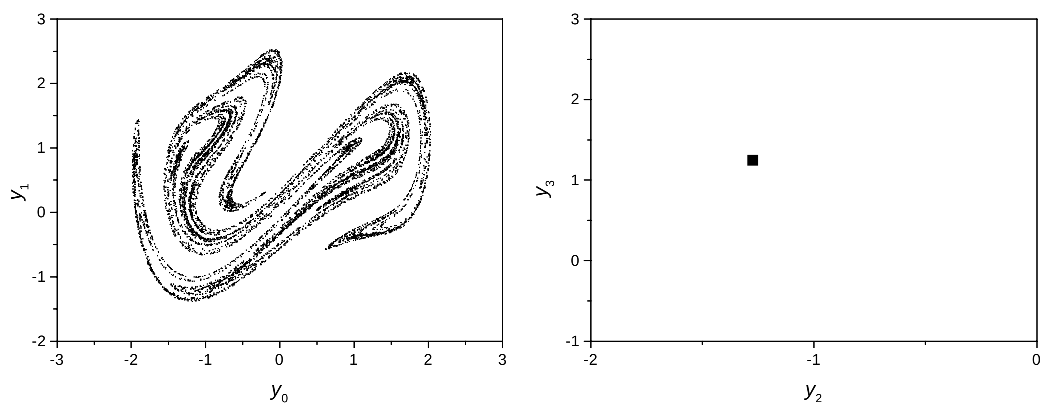

Figure 2 - Poincaré section for two oscillators ($k_2 = c_2 = 0$).

In order to start the analysis of transmissibility of motion, a linear spring connection is considered. Therefore, $c_2 = 0$, and the parameter k_2 may vary. First, one considers bifurcation diagrams which represent the stroboscopically sampled displacement values, y_0 and y_2, under the slow quasi-static increase of parameter k_2 (Fig.3). The chaotic motion of mass m_1 is completely transmitted to mass m_2 when k_2 decreases, however, the Poincaré section associated with mass m_2 has a different pattern from the usual form of the strange attractor presented by the mass m_1 (Fig.4).

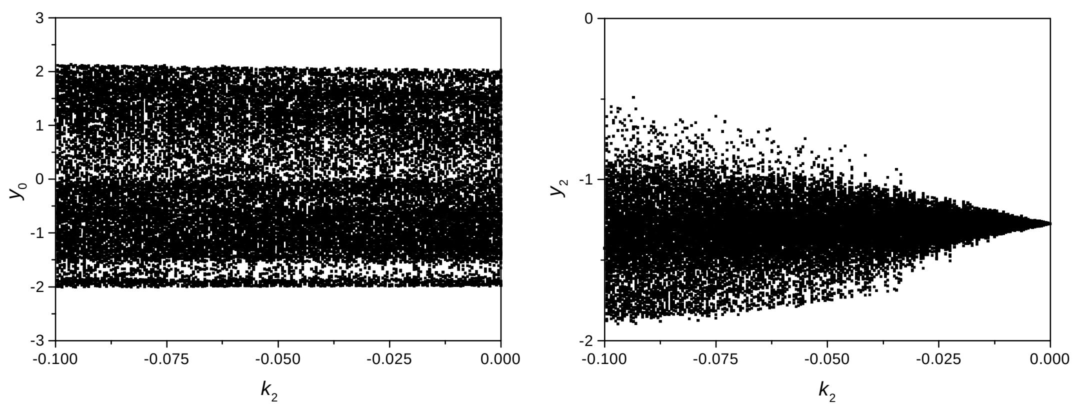

Figure 3 - Bifurcation diagrams for y_0 vs k_2 and y_2 vs k_2 with $c_2 = 0$.

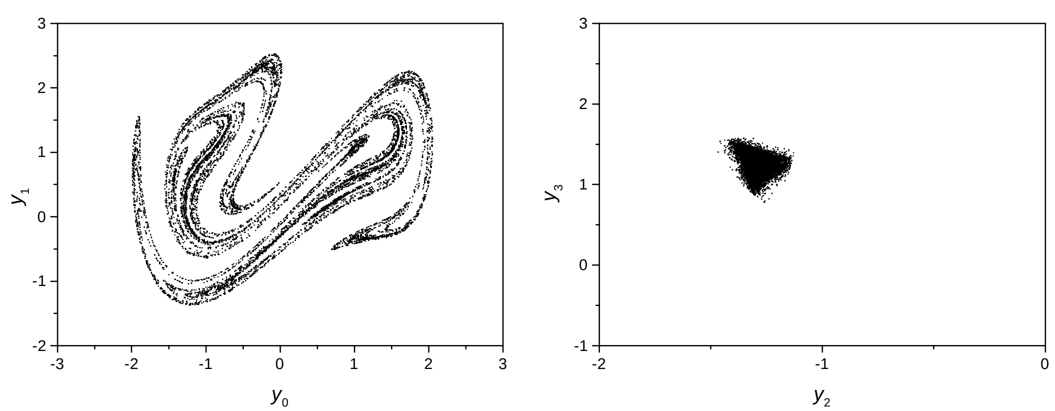

Figure 4 - Poincaré section with $c_2 = 0$ and $k_2 = -0.02$.

Now, a linear viscous damper connection is focused. Hence, $k_2 = 0$, and the parameter c_2 is used to analyze the transmissibility of motion between the two oscillators. One considers bifurcation diagrams relating the sampled displacement values, y_0 and y_2, under the slow quasi-static increase of parameter c_2 (Fig.5). The energy dissipation on the connection establishes a different kind of transmissibility. For high values of the parameter c_2, the motion of the two masses tends to be similar.

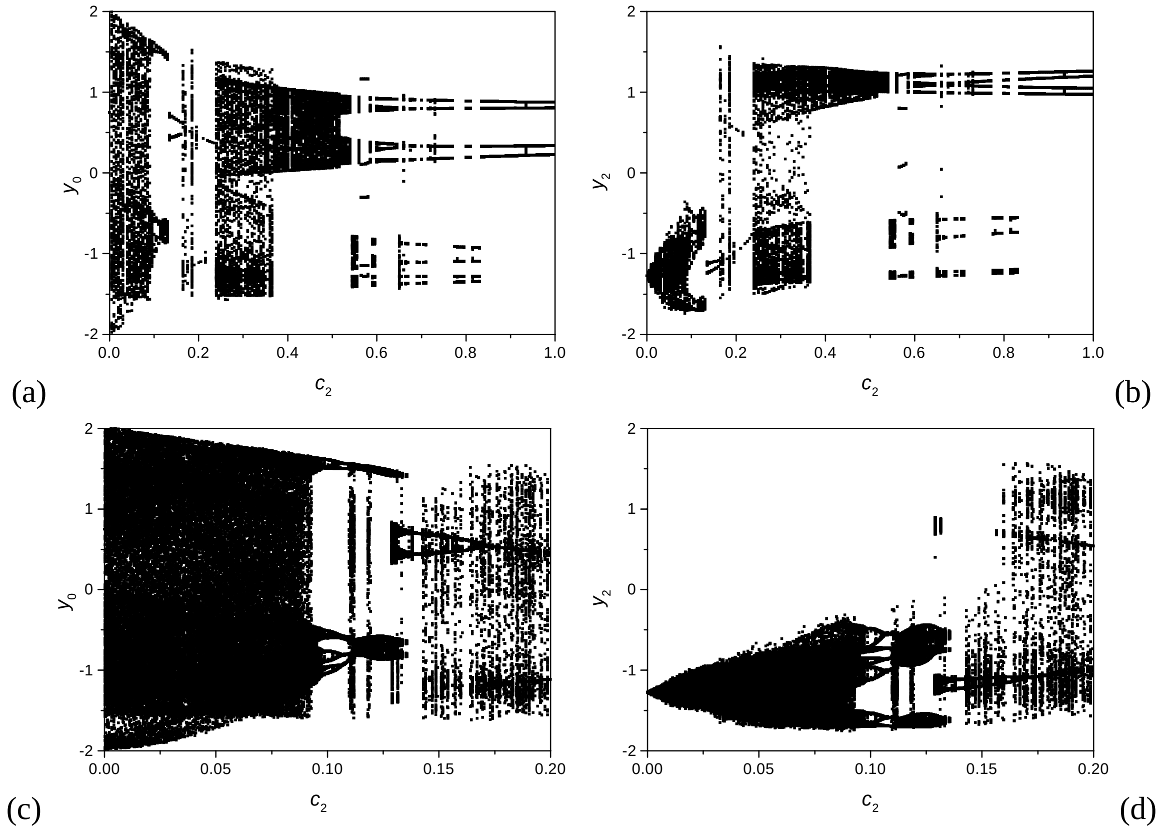

Figure 5 - Bifurcation diagrams with $k_2 = 0$. (a) y_0 vs c_2 ; (b) y_2 vs c_2;
(c) Zoom for y_0 vs c_2; (d) Zoom for y_2 vs c_2.

Figures 6-7 show the Poincaré sections for $k_2 = 0$ and some different values of the dissipation parameter. Figure 6 considers $c_2 = 0.05$, and chaotic motion is observed for both masses. Again, the strange attractor of mass m_1 has a usual form, while mass m_2 presents a different pattern. On the other hand, Fig.7 considers $c_2 = 0.1$, and there is a periodic motion.

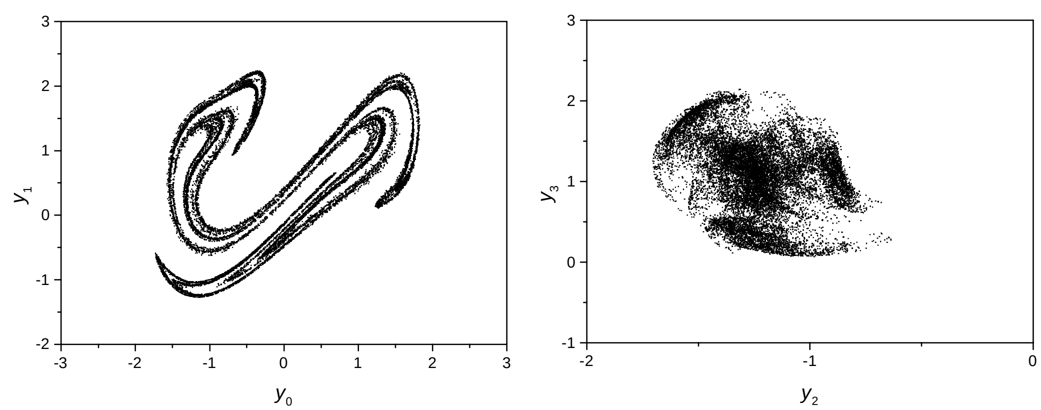

Figure 6 - Poincaré section with $k_2 = 0$ and $c_2 = 0.05$.

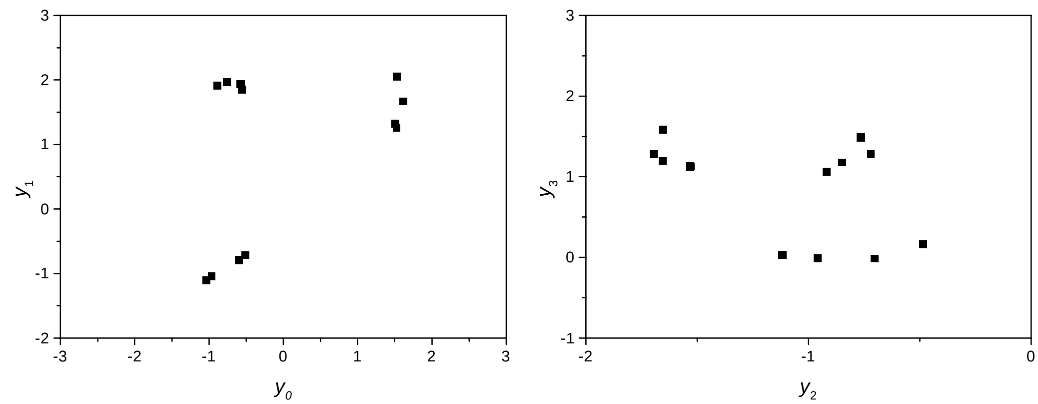

Figure 7 - Poincaré section with $k_2 = 0$ and $c_2 = 0.1$.

A linear spring-viscous damper connection is now conceived. The spring constant is $k_2 = -0.02$, and the parameter c_2 is used to analyze the transmissibility of motion between the two oscillators. Figure 8 shows the bifurcation diagrams for this situation.

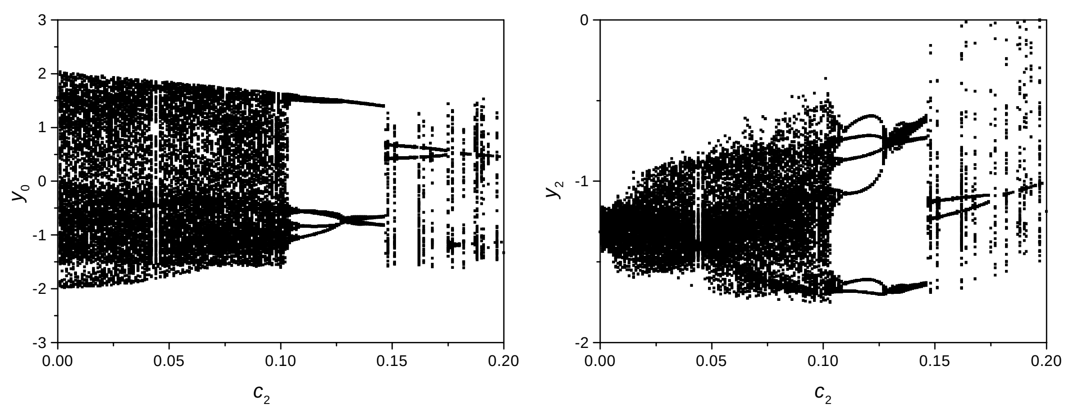

Figure 8 - Bifurcation diagrams for y_0 vs c_2 and y_2 vs c_2 with $k_2 = -0.02$.

Figures 9-10 show the Poincaré sections for $k_2 = -0.02$ and some different values of the dissipation parameter. Figure 9 considers $c_2 = 0.05$, while Figure 10 considers $c_2 = 0.1$. In these examples, chaotic motion is observed for both masses, and again, the Poincaré section associated with mass m_2 present a different pattern of the strange attractor.

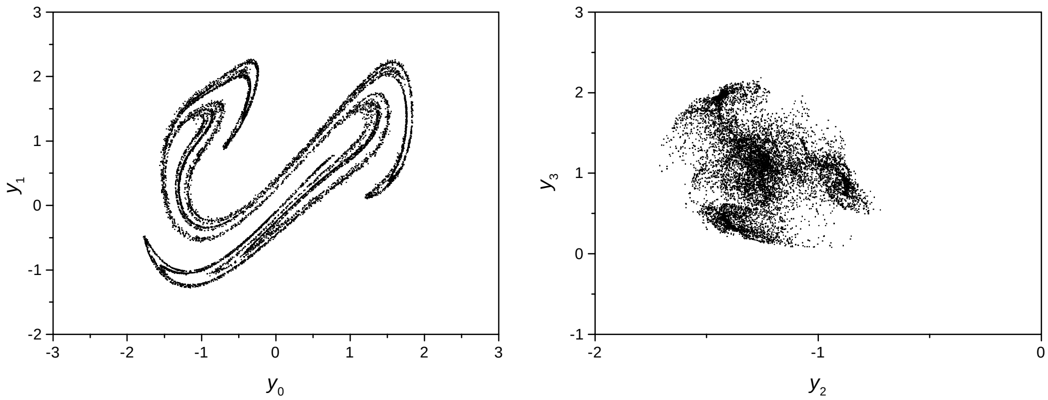

Figure 9 - Poincaré section with k_2 = -0.02 and c_2 = 0.05.

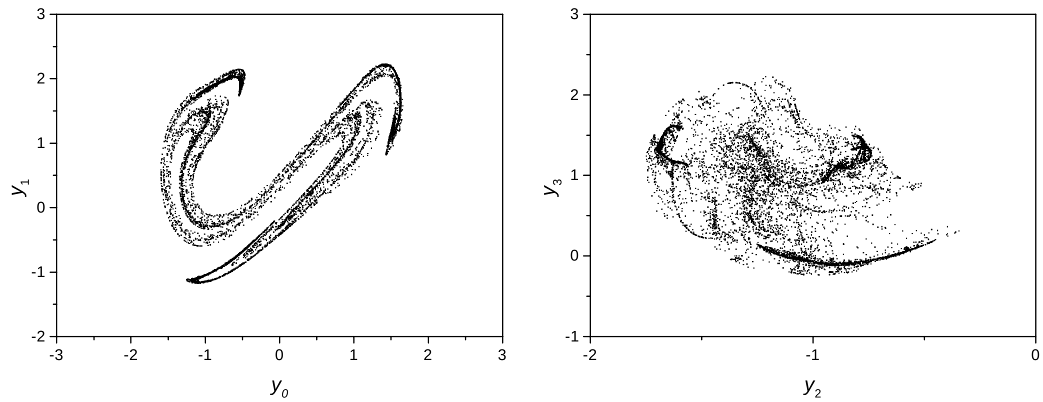

Figure 10 - Poincaré section with k_2 = -0.02 and c_2 = 0.1.

4. NON-LINEAR CONNECTION

In this section, a non-linear connection between the two Duffing oscillators, both with one-degree of freedom, is investigated. Therefore, one considers a_2 = 1, and the parameters c_2 e k_2 are used to analyze the transmissibility of motion between the two oscillators. The same conditions presented on the preceding section are taken. In order to start the analysis, one considers bifurcation diagrams relating the sampled displacement values, y_0 and y_2, under the slow quasi-static increase of parameter k_2. It is also assumed that there is no dissipation on the connection, c_2 = 0 (Fig.11).

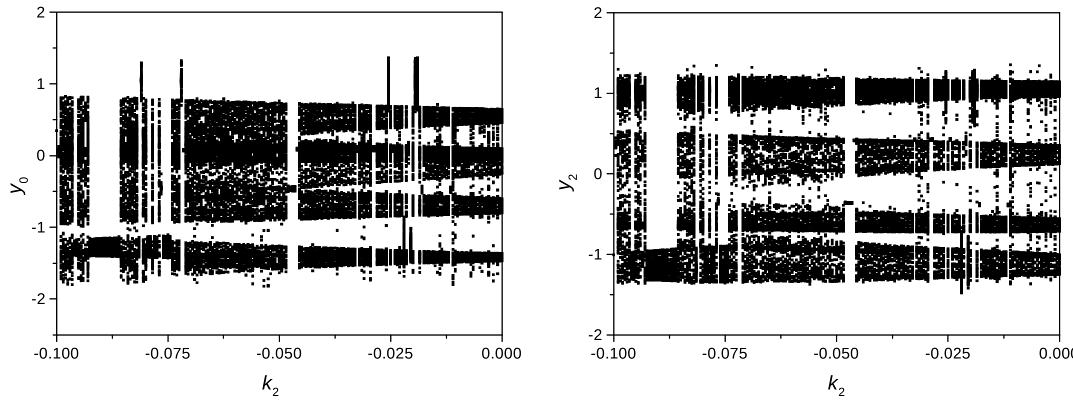

Figure 11 - Bifurcation diagrams for y_0 vs k_2 and y_2 vs k_2 with a_2 = 1 and c_2 = 0.

Figures 12-13 show the Poincaré sections for two different sets of parameters. For these examples, a different pattern of chaotic motion occurs for the two masses.

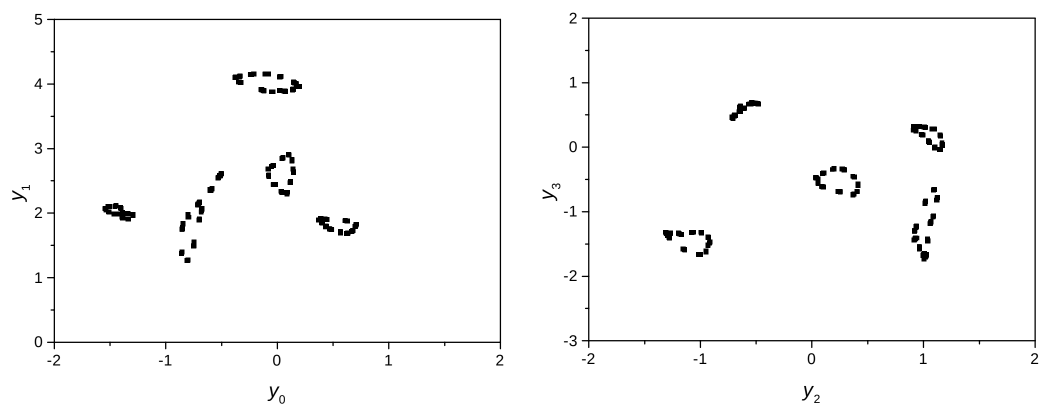

Figure 12 - Poincaré sections with a_2 = 1, c_2 = 0 and k_2 = -0.0375.

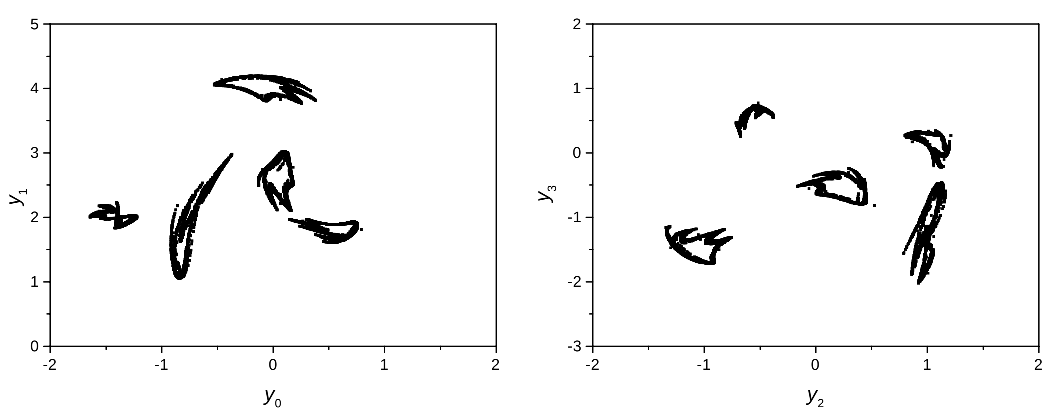

Figure 13 - Poincaré sections with a_2 = 1, c_2 = 0 and k_2 = -0.0625.

A non-linear spring-viscous damper connection is in order. The spring constant is k_2 = -0.02, and the parameter c_2 is used to analyze the transmissibility of motion between the two oscillators. Figure 14 shows the bifurcation diagrams for this situation.

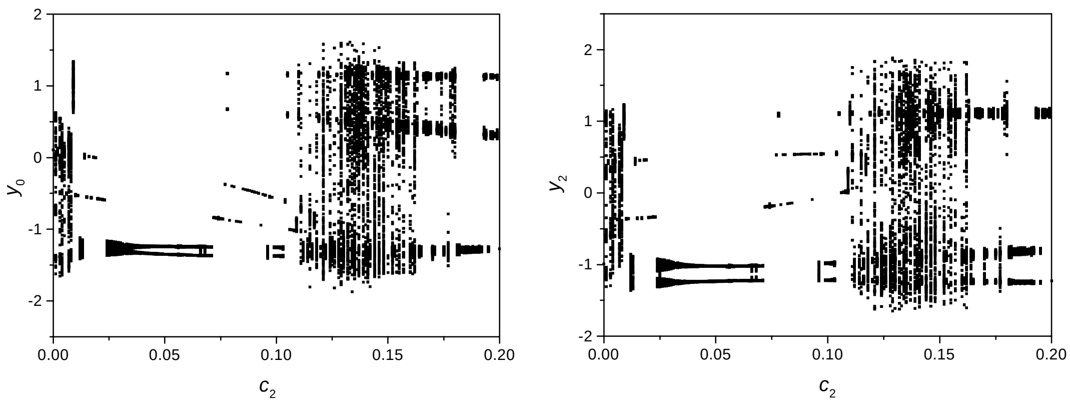

Figure 14 - Bifurcation diagrams for y_0 vs c_2 and y_2 vs c_2 with a_2 = 1 and k_2 = -0.02.

By considering a_2 = 1, k_2 = -0.02 and c_2 = 0.05, a period-2 response occurs. Figure 15 shows a typical Poincaré section for this situation.

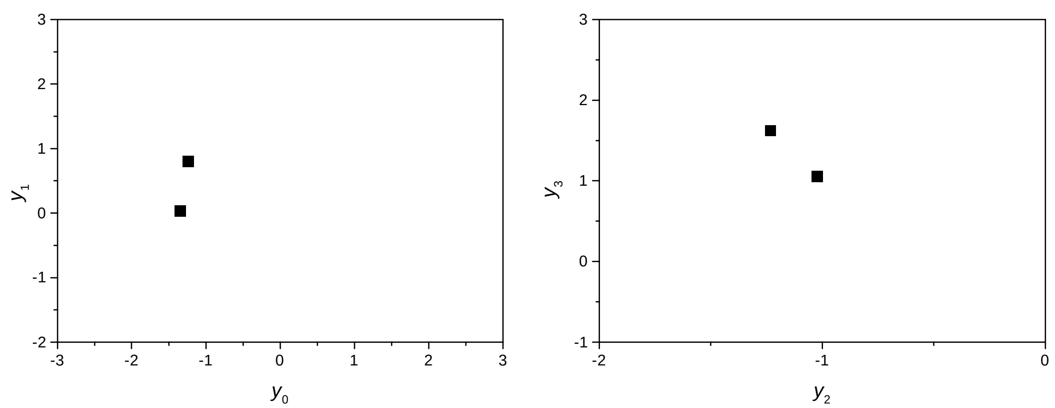

Figure 15 - Poincaré section with a_2 = 1, k_2 = -0.02 and c_2 = 0.05.

5. CONCLUSIONS

This contribution discusses the chaotic response of a two-degree of freedom Duffing oscillator. Numerical simulations are obtained using the fourth order Runge-Kutta method for numerical integration. Since the equations of motion are associated with a four-dimensional system, the analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, connected by a spring-dashpot system. With this assumption, it is possible to analyze the transmissibility of motion between the two oscillators. The results show that chaotic motion of one mass is transmitted with a different pattern to the other mass and reveals that very complex behavior can be expected for other dynamical system either with multiple degrees of freedom or continuous.

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